

Characterisation of the parameters of maximum weight spectrum codes according to their spread

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Abstract

We introduce the concept of spread of a code, and we specialize it to the case of maximum weight spectrum (MWS) codes. We classify two newly-defined sub-families of MWS codes according to their weight distributions, and completely describe their fundamental parameters. We focus on one of these families, the strictly compact MWS codes, proving their optimality as MWS codes and linking them to known codes.

1 Introduction

Let \mathbb{F}_q be a finite field with q elements. A $[n, k]_q$ linear code is a k -dimensional subspace of \mathbb{F}_q^n . The (Hamming) weight $w(x)$ of a vector $x \in \mathbb{F}_q^n$ is defined to be the number of nonzero components of x . The minimum of weights where $x \neq 0$ is the minimal distance d of the code. A linear code $[n, k]_q$ whose the minimum distance is d shall be denoted by $[n, k, d]_q$. A generator matrix for an $[n, k]_q$ linear code C is a $k \times n$ matrix over \mathbb{F}_q whose row vectors generate C . Let $(n + 1)$ -tuples $\{A_0, A_1, \dots, A_n\}$ be the weight distribution of an $[n, k, d]_q$ linear code C where A_i is the number of codewords in C with weight i . We denote by $S(C)$ the set of non-zero weights of a linear code C , i.e. $s \in S(C)$ if there exists $c \in C \setminus \{0\}$ such that $w(c) = s$.

The weight distribution of linear codes has been appeared in many works over the years. MacWilliams in [7] constructed linear codes by employing the elements of a given weight set. She also proved that the weight set of a linear code can be derived from the weight set of its dual. Delsarte in [4] discussed some properties of codes using their weight distributions. In particular an upper bound

$$|C| \leq \sum_{i=0}^r \binom{n}{i} (q-1)^i$$

on the size of a q -ary code C with r nonzero distinct weights was proposed. In addition, a lower bound on the size of C was obtained using the number of distinct nonzero weights of its dual code C^\perp . More bounds on $|C|$ can be found in [5].

The concept of binary linear codes with distinct weights has been firstly proposed and fully characterised in [6]. In this work, the authors define a *distinct weight* (DW) code as a binary code whose weight distribution is of the form $A_i \in \{0, 1\}$ for each i . Other than providing an infinite family of DW codes they study the automorphism group of DW codes.

Definition 1. We denote by B_k the $[2^k - 1, k, 1]_2$ DW code defined by the generator matrix

$$G = \begin{bmatrix} 1_1^{(k)} \\ 1_2^{(k)} \\ \vdots \\ 1_k^{(k)} \end{bmatrix}$$

where $1_i^{(k)}$ for $i = 1, \dots, k$ is the $(2^k - 1)$ -bits row vector whose first $2^i - 1$ bits are equal to 1 and the remaining are equal to 0, e.g.

$$\begin{cases} 1_1^{(3)} = (1, 0, 0, 0, 0, 0, 0) \\ 1_2^{(3)} = (1, 1, 1, 0, 0, 0, 0) \\ 1_3^{(3)} = (1, 1, 1, 1, 1, 1, 1) \end{cases} \Rightarrow \begin{bmatrix} 1_1^{(3)} \\ 1_2^{(3)} \\ 1_3^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Due to [6], there exist binary linear codes in which no two distinct codewords have the same Hamming weight and they are called distinct weight codes. Recently, independently from [6], the authors of [11] proposed the problem of determining the maximum possible number of distinct weights in a block code over any finite field \mathbb{F}_q . Other than providing a complete solution in the general nonlinear q -ary case, they showed a construction for binary linear codes attaining the maximum possible number of distinct weights. This family coincides with the family B_k recalled in Definition 1. In the same work, the authors proposed a bound and conjectured that the maximum number of distinct nonzero weights that a k -dimensional q -ary code can have is $\frac{q^k - 1}{q - 1}$. This work opened several lines of research, starting from two independent works providing solutions for the conjecture [2], [8]. Following the notation established in [2], the codes that attain this upper bound are now known as *maximum weight spectrum* (MWS) codes. Observe that DW codes are a particular case of MWS codes. Shorter codes with maximum number of weights for a given dimension were later discussed in [1, 3]. In [10], the authors discussed the largest number of nonzero weights a cyclic code of dimension k over \mathbb{F}_q can have.

Theorem 2. [11] *Let C be a k -dimensional q -ary linear code. Then the maximum possible number q_k of distinct non-zero weights in C equals $q_k = \frac{q^k - 1}{q - 1}$.*

Definition 3. [2] An $[n, k]_q$ code C such that $|S(C)| = \frac{q^k - 1}{q - 1}$ is called *maximum weight spectrum* (MWS) codes.

We recall another important example of MWS code.

Definition 4. We denote with D_q the $[\frac{q(q+1)}{2}, 2, \frac{q(q-1)}{2}]_q$ code generated by

$$G' = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \underbrace{\alpha^1}_1 & \underbrace{\alpha^2}_2 & \alpha^2 & \cdots & \underbrace{\alpha^{q-1}}_{q-1 \text{ times}} & \cdots & \alpha^{q-1} & \underbrace{0}_{q \text{ times}} & \cdots & 0 \end{bmatrix},$$

where α is a primitive element of \mathbb{F}_q .

Observe that for $k = 2$ and $q = 2$, the codes B_k and D_q coincide. This is a hint that there exists a family of MWS codes sharing the same structure and properties which will be named as *strictly compact* MWS codes and denoted by $\mathcal{SC}_{q,k}$ later.

Subsequent works propose several constructions for new families of codes [2, 3, 9], trying in particular to obtain tight bounds on the minimum possible length of MWS codes. Another related open question is whether, given two integers k and s , there exists a linear code with dimension k and exactly s distinct weights. A positive answer has been provided in [8] for the particular case of binary codes, while the q -ary case is left as a conjecture.

Due to [2], the minimum possible length of an $[n, k, d]_q$ MWS code is $n = \frac{q}{2} \cdot \frac{q^k - 1}{q - 1}$. An $[n, k, d]_q$ MWS code achieves this length if it has codewords of every Hamming weight from d to n and we call it *strictly compact* (Definition 6) MWS code. An $[m, l, s]_q$ *compact* (Definition 6) MWS code has codewords of every Hamming weight from s to $s + \frac{q^l - 1}{q - 1} - 1$.

In this work we investigate the parameters of MWS codes according to new classifications. The properties and parameters of MWS codes, strictly compact MWS codes and compact MWS codes are discussed. In particular, the codes B_k and D_q are strictly compact, and, as all strictly compact codes, they are optimal MWS codes (we will prove this in Corollary 14). We introduce the concept of *spread* (Definition 8) of a code, a sort of distance between the weight distribution of a code from a reference one. We use strictly compact codes as a reference code to measure the spread of MWS codes, namely, the spread of a MWS code C is the distance between its weight distribution and the one of a hypothetical strictly compact MWS code with equal length (see Definition 9). This choice is based upon the optimality of strictly compact codes, and it turns out that the spread is deeply linked with the length of MWS. Moreover, we investigate bounds on the minimum distance and the spread of known MWS codes.

2 Notation and remarks

In this section we discuss the parameters of MWS codes according to their weight distribution. In particular we will see how length, dimension and minimum distance of MWS codes are related to their weight distribution. Particular emphasis will be put on the parameters of the family $\mathcal{SC}_{q,k}$ briefly introduced in Section 1. The following lemma is a direct consequence of the definition of linear codes, hence we recall it without providing a proof.

Lemma 5. Let $\{A_i\}_{i \in \{0, \dots, n\}}$ be the weight distribution of an $[n, k, d]_q$ linear code C . If C has no coordinate position is identically 0 (non-degenerate), then

$$n = \frac{\sum_i A_i i}{q^k - q^{k-1}}.$$

We introduce some useful definitions to classify MWS codes according to their weight distribution.

Definition 6. An MWS $[n, k, d]_q$ code C is *compact* if its set of weights $S(C)$ is $S(C) = \{d, d + 1, \dots, d + q_k - 1\}$ where $q_k = \frac{q^k - 1}{q - 1}$, and it is *strictly compact* if it is compact and $n \in S(C)$. $\mathcal{SC}_{q,k}$ denotes a strictly compact MWS code with dimension k over \mathbb{F}_q .

Proposition 7. B_k and D_q in Definitions 1 and 4 are strictly compact for respectively any choice of k and q . In particular

$$S(B_k) = \{1, \dots, n\} = \underbrace{\{1, 2, \dots, 2^k - 2, 2^k - 1\}}_{q_k \text{ consecutive integers}},$$

and

$$S(D_q) = \{d, \dots, n\} = \left\{ \underbrace{\left\{ \frac{q(q-1)}{2}, \dots, \frac{q(q+1)}{2} \right\}}_{q_k \text{ consecutive integers}} \right\}.$$

Proof. It follows from computation. □

Equivalent to Definition 6, an MWS is compact if its weight distribution is

$$\{A_i\}_{i \in \{0, \dots, n\}} = \left\{ 1, \underbrace{0, \dots, 0}_{1 \leq i \leq d-1}, \underbrace{q-1, \dots, q-1}_{d \leq i \leq d+q_k-1}, \underbrace{0, \dots, 0}_{i > d+q_k-1} \right\}$$

while it is strictly compact if

$$\{A_i\}_{i \in \{0, \dots, n\}} = \left\{ 1, \underbrace{0, \dots, 0}_{1 \leq i \leq d-1}, \underbrace{q-1, \dots, q-1}_{i \geq d} \right\}.$$

Observe that for a strictly compact MWS we can write

$$S(C) = \{d, d + 1, \dots, n\} = \{n - q_k + 1, \dots, n - 1, n\} = \{n - i \mid i = 0, 1, \dots, q_k - 1\},$$

and in the general case, we can write the set of weights of an MWS code as

$$S(C) = \{n - s_0, \dots, n - s_{q_k-1}\}. \tag{1}$$

A strictly compact MWS code with $d = 1$ is called *full weight spectrum* (FWS) code in [1].

Definition 8. Let C and \bar{C} be two q -ary codes with the same length n and dimension k . We define the *spread* of C w.r.t a target code \bar{C} as the value

$$\Delta_{\bar{C}}(C) = \frac{1}{q-1} \sum_{i=1}^M (\bar{w}_i - w_i) ,$$

where $\{w_1, \dots, w_M\}$ and $\{\bar{w}_1, \dots, \bar{w}_M\}$ are the multiset of all non-zero weights of C and \bar{C} .

We remark that $\Delta_{\bar{C}}(C)$ is therefore equal to $\frac{1}{q-1} \left(\sum_{i=1}^n i \bar{A}_i - \sum_{j=1}^n j A_j \right)$, with \bar{A}_i and A_j the weight distributions of \bar{C} and C respectively. Observe that $\Delta_{\bar{C}}(C) = -\Delta_C(\bar{C})$. For our purposes, we specialize Definition 8 to the case of MWS codes, using a hypothetical strictly compact MWS code as the target code. In this case we can therefore omit to specify the target code, since given the $[n, k]_q$ MWS code C we know exactly the weight distribution of a strictly compact $[n, k]_q$ MWS code.

Definition 9. The *spread* of an MWS code C with $S(C) = \{n - s_0, \dots, n - s_{q_k-1}\}$ is the value

$$\Delta(C) = (s_0 - 0) + (s_1 - 1) + \dots + (s_{q_k-1} - q_k + 1) = \sum_{i=0}^{q_k-1} (s_i - i). \quad (2)$$

With this definition, $\Delta(C)$ can be thought as a measure of how much the weight distribution of an MWS code is spread across the entire set $\{1, \dots, n\}$, in terms of the distance from the weight distribution of a hypothetical strictly compact MWS code of length n . Due to Definition 9, we can equivalently define a strictly compact MWS code C as an MWS code with spread $\Delta(C) = 0$. As we will see, strictly compact MWS codes are optimal codes, namely, there exist no MWS code with equal dimension and length strictly less than their length. To prove this claim, the next section deals with the characterisation of the parameters of strictly compact MWS codes and their comparison with general MWS codes.

3 Strictly Compact MWS codes

In this section we use the notation introduced in previous section to discuss the parameters of MWS codes, with a focus on strictly compact codes. We prove in particular the link between length and spread of MWS codes, implying the optimality of strictly compact codes.

Theorem 10. Let $\mathcal{SC}_{q,k}$ be a strictly compact MWS code of dimension k over \mathbb{F}_q . Then its parameters are

$$\left[\frac{q}{2} q_k, k, \left(\frac{q}{2} - 1 \right) q_k + 1 \right]_q .$$

Proof. A strictly compact $[n, k, d]_q$ MWS code C is by definition a code with spread $\Delta(C) = 0$. In particular

$$S(C) = \{n, n-1, \dots, n - q_k + 1\}, \quad \text{and} \quad A_i = q - 1, i \in S(C). \quad (3)$$

By Lemma 5 we have $n = \frac{\sum_i A_i i}{q^k - q^{k-1}}$, which, due to Equation (3), can be written as

$$n = \frac{\sum_{i=0}^{q_k-1} (q-1)(n-i)}{q^k - q^{k-1}} = \frac{n \cdot q_k - \sum_{i=0}^{q_k-1} i}{q^{k-1}} = \frac{n \cdot q_k - \frac{(q_k-1)q_k}{2}}{q^{k-1}},$$

hence

$$n (q_k - q^{k-1}) = \frac{(q_k - 1) q_k}{2} \tag{4}$$

We observe that on the left-hand side of Equation (4), the coefficient of n is $q_k - q^{k-1} = q_{k-1}$, while on the right-hand side we have $q_k - 1 = q \cdot q_{k-1}$. By substitution, we deduce

$$n q_{k-1} = \frac{(q \cdot q_{k-1}) q_k}{2}, \tag{5}$$

which leads to the claimed length of C .

The proof that $d = (\frac{q}{2} - 1) q_k + 1$ is then a direct consequence of C being a strictly compact MWS code, since in this case the minimum weight is $n - q_k + 1$. \square

Example 11. As expected, if we use $q = 2$, then the parameters are $[2^k - 1, k, 1]_2$, namely the parameters of the known code $\mathcal{SC}_{2,k}$. Similarly, if we use $k = 2$, the parameters listed in Theorem 10 become $[\frac{q}{2} q_2, 2, (\frac{q}{2} - 1) q_2 + 1]_q = [\frac{q}{2} (q + 1), 2, \frac{q}{2}(q - 1)]_q$, i.e. the parameters of $\mathcal{SC}_{q,2}$.

Example 12. Consider $q = 4$ and $k = 3$, namely the smallest case with q even which are not covered by $\mathcal{SC}_{2,k}$ or $\mathcal{SC}_{q,2}$. If a strictly compact MWS code would exist, then it would be a $[42, 3, 22]_4$ code.

Theorem 13. *Let C be an MWS code. Then $n = \frac{q}{2} q_k + \frac{\Delta(C)}{q_{k-1}}$.*

Proof. We proceed similarly to the proof of Theorem 10, where Equation (3) is substituted by Equation (1). This implies, after some computation, that Equation (4) becomes

$$n (q_k - q^{k-1}) = \frac{(q_k - 1) q_k}{2} + \sum_{i=0}^{q_k-1} (s_i - i).$$

We recall that by Equation (2) the sum on the right-hand side is $\Delta(C)$. In this way, instead of Equation (5) we have $n q_{k-1} = \frac{(q \cdot q_{k-1}) q_k}{2} + \Delta(C)$. \square

Corollary 14. *Strictly compact MWS codes are optimal among MWS codes.*

Proof. It follows from Theorem 10 and Theorem 13. The spread of an MWS code is indeed equal to zero if and only if the code is strictly compact, and the length of an MWS code grows together with its spread. \square

4 On the parameters of MWS codes

In this section we consider again general MWS codes, and we use strictly compact MWS codes to obtain a characterization of their parameters.

Corollary 15. *Let C be an MWS code with q odd and odd dimension k . Then $\Delta(C) > 0$. In particular, there does not exist a strictly compact MWS code $\mathcal{SC}_{q,k}$ for q and k both odd.*

Proof. If $\Delta(C) = 0$ then C is a strictly compact MWS code, hence by Theorem 10 we know that its length is $n = \frac{q}{2}q_k \in \mathbb{Z}$. We have two possible cases: either q is even, or q_k is. Observe that if q is odd, the latter is true if and only if k is even. \square

Corollary 16. *Let C be an $[n, k, d]_q$ MWS code.*

1. *Let $q \cdot k$ be even. Then $\Delta(C) = hq_{k-1}$, with h a non-negative integer.*
2. *Let $q \cdot k$ be odd. Then $\Delta(C) = \frac{2h+1}{2}q_{k-1}$, for some positive integer h .*

Proof. The length of C is equal to $\frac{q}{2}q_k + \frac{\Delta(C)}{q_{k-1}} \in \mathbb{Z}$.

1. If $q \cdot k$ is even, then either q is even or q_k is. This implies that $\frac{q}{2}q_k \in \mathbb{Z}$, and since $n \in \mathbb{Z}$, so has to be $n - \frac{q}{2}q_k = \frac{\Delta(C)}{q_{k-1}}$.
2. If both q and k are odd, then $\frac{q}{2}q_k$ is not an integer. However, since $n \in \mathbb{Z}$, $\frac{\Delta(C)}{q_{k-1}}$ is equal to $\frac{2h+1}{2}$ for a positive integer h . This implies that the spread of C is $\Delta(C) = \frac{2h+1}{2}q_{k-1}$.

\square

Example 17. Consider an MWS code C of dimension $k = 3$ over \mathbb{F}_3 , so that by Corollary 16 C has spread $\Delta(C) = \frac{2h+1}{2}q_{k-1} = 2(2h+1) \geq 6$. If we assume $\Delta(C) = 6$ then we can apply Theorem 13 to deduce that the length of C is

$$n = \frac{q}{2} \cdot \frac{q^k - 1}{q - 1} + \frac{q - 1}{q^{k-1} - 1} \cdot 6 = \frac{3}{2} \cdot \frac{3^3 - 1}{2} + \frac{2}{3^2 - 1} \cdot 6 = \frac{3}{2} \cdot \frac{26}{2} + \frac{3}{2} = \frac{39}{2} + \frac{3}{2} = 21.$$

Some possible sets of non-zero weights of such an MWS code are

$$\begin{aligned} &\{3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\} \\ &\{4, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\} \\ &\{5, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\} \\ &\{6, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\} \\ &\{7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}. \end{aligned}$$

Proposition 18. *Let C be an $[n, k, d]_q$ MWS code.*

1. *If either q or k are even, then*

$$\begin{cases} n = \frac{q}{2}q_k + h \\ d \geq q_k(\frac{q}{2} - 1) + h(\frac{q^{k-1}}{q_k} - q_{k-1}) + 1 \\ d \leq q_k(\frac{q}{2} - 1) + h\frac{q^{k-1}}{q_k} + 1 \end{cases}$$

where h is a non-negative integer.

2. *If both q and k are odd, then*

$$\begin{cases} n = \frac{q}{2}q_k + h + \frac{1}{2} \\ d \geq q_k(\frac{q}{2} - 1) + (\frac{2h+1}{2}q_{k-1})(\frac{1}{q_k} - 1) + 1 \\ d \leq q_k(\frac{q}{2} - 1) + (\frac{2h+1}{2})(\frac{q^{k-1}}{q_k}) + 1 \end{cases}$$

where h is a positive integer.

Proof. It follows from Corollary 16, noticing that the minimum distance is linked to the spread of the code, which in turn depends on h . \square

5 Compact MWS codes

In this section we focus on the parameters of compact MWS codes, which are MWS codes whose set of weights is of the form $\{d, d + 1, \dots, d + q_k - 1\}$.

Let us start with considering a code with either q odd and k even or with q even, i.e. as in Corollary 16, case 1, so that the spread is hq_{k-1} and the length is $n = \frac{q}{2}q_k + h$.

If we suppose that $A_n = 0$, then the spread has to be at least q_k , and this implies $h \geq \frac{q_k}{q_{k-1}}$. As a consequence $n \geq \frac{q}{2}q_k + \frac{q_k}{q_{k-1}}$.

Suppose now the maximum weight in C is $n - j$, namely $A_{n-j} \neq 0$ and $A_i = 0$ for any $i > n - j$. Similarly to above, $h \geq j\frac{q_k}{q_{k-1}}$. In this case we have obtained that $n \geq \frac{q}{2}q_k + j\frac{q_k}{q_{k-1}}$. We obtain the following complete characterization of the parameters of compact MWS codes.

Corollary 19. *Consider an $[n, k, d]_q$ compact MWS code C . Then*

1. *if $q \cdot k$ is even, then the parameters of C are*

$$\left[\left(\frac{q}{2} + \frac{j}{q_{k-1}} \right) q_k, k, q_k \left(\frac{q}{2} - 1 \right) + j \left(\frac{q_k}{q_{k-1}} - 1 \right) + 1 \right],$$

where $q_{k-1} | j$.

2. *if $q \cdot k$ is odd, then the parameters of C are*

$$\left[\left(\frac{q}{2} + \frac{j}{q_{k-1}} \right) q_k, k, q_k \left(\frac{q}{2} - 1 \right) + j \left(\frac{q_k}{q_{k-1}} - 1 \right) + 1 \right],$$

where $j = \frac{2r+1}{2}q_{k-1}$ and $r \in \mathbb{N}$.

Proof. If $q \cdot k$ is even, which was already introduced in the discussion above, we consider a code with maximum weight $n - j$, and we assume it is compact. Then the length is $n = \frac{q}{2}q_k + j\frac{q_k}{q_{k-1}}$ and therefore the distance is

$$d = n - j - q_k + 1 = \frac{q}{2}q_k + j\frac{q_k}{q_{k-1}} - j - q_k + 1 = q_k\left(\frac{q}{2} - 1\right) + j\left(\frac{q_k}{q_{k-1}} - 1\right) + 1$$

The case in which $q \cdot k$ is odd, namely as in Corollary 16, case 2, is very similar, since if the maximum weight is $n - j$, the spread is $hq_{k-1} + \frac{1}{2} \geq jq_k$, then $h \geq j\frac{q_k}{q_{k-1}} - \frac{1}{2}$. As a consequence, the length is $n \geq \frac{q}{2}q_k + j\frac{q_k}{q_{k-1}} + \frac{1}{2}$: which coincides with the bound in the first case and this bound is attained with equality if the code is compact. \square

As a consequence of Corollary 19 we have the following result.

Corollary 20. *The only compact MWS code with $d = 1$ is the binary full weight spectrum code.*

6 Known codes

MWS codes have been studied in [2, 8], where the authors also presented some bounds on the length of $[n, k]_q$ MWS codes. In this section we investigate bounds on the minimum distance and the spread of known MWS $[n, k, d]_q$ codes. In this section, by vector $A = [a_1, \dots, a_k]$ we mean a nonzero element of \mathbb{F}_q^k up to projective equivalence. We only consider *non-degenerate* $[n, k]_q$ codes. Let us first give some definitions and a simple observation.

Definition 21. Let m_1, \dots, m_k be elements in \mathbb{F}_q not all equal to zero. The set \mathcal{H} consisting of all vectors $X = [x_1, \dots, x_k]$ such that

$$m_1x_1 + \dots + m_kx_k = c \quad , \quad \text{for } c \in \mathbb{F}_q \quad ,$$

is called a *hyperplane*, which is a $(k - 1)$ -dimensional subspace of $(\mathbb{F}_q)^k$.

The number of the 1-dimensional vector spaces in $(\mathbb{F}_q)^k$ is equal to $\frac{q^k - 1}{q - 1} = q_k$ which coincides with the number of $(k - 1)$ -dimensional subspaces (hyperplanes) of $(\mathbb{F}_q)^k$. Consequently, every hyperplane contains $\frac{q^{k-1} - 1}{q - 1} = q_{k-1}$ k -vectors over \mathbb{F}_q . Any pair of distinct hyperplanes in $(\mathbb{F}_q)^k$ intersects in a $(k - 2)$ -dimensional subspace over \mathbb{F}_q .

Definition 22. Let C be an $[n, k]_q$ code with generator matrix G where M is the (multi)set of columns of G and A is a subspace in $(\mathbb{F}_q)^k$. Then $\text{Char}_G(A)$ is the number, including multiplicity, of k -vectors in the (multi)set $M \cap A$. We denote by $m(v)$ the multiplicity of the vector v in M .

Remark 23. Let G be a generating matrix of an $[n, k]_q$ code C . For any non-zero vector $m = (m_1, \dots, m_k) \in (\mathbb{F}_q)^k$, the hyperplane $m_1x_1 + \dots + m_kx_k = 0$ contains $n - s$ columns (with multiplicity) of G if and only if the codeword mG has weight s . So we have a hyperplane with $\text{Char}_G(H) = n - s$ if and only if there is a codeword $c \in C$ with weight s .

Theorem 24. *There exists an $[n, k, d]_q$ MWS code for each prime power q and $k \geq 2$, where*

$$n = 2^{qk} - 1 \quad \text{and} \quad d = 2^{q^{k-1}-1} - 1.$$

Proof. The geometric construction given in [2, Theorem 3.4] leads to an $[n, k, d]_q$ code of length $n = 2^{qk} - 1$. In the proof the characters of different hyperplanes are ranged from $2^{qk-1} - 1$ to $2^{qk} - 2^{q^{k-1}-1}$. So the minimum distance is $n - 2^{qk} + 2^{q^{k-1}-1} = 2^{q^{k-1}-1} - 1$. \square

Lemma 25. *If C is an $[n, k, d]_q$ MWS code with $k \geq 2$, then*

$$n \geq \lceil \frac{q}{2} q_k \rceil \quad , \quad d \geq \lceil \frac{q}{2} q_k - q_k + 1 \rceil \quad \text{and} \quad \Delta(C) \geq 0.$$

Proof. The lower bound for n was proven in [2, Lemma 5.1] and also can be seen as a consequence of Theorem 13 and the bound for minimum distance is already given in Proposition 18. Finally the bound for $\Delta(C)$ comes from its definition. \square

Proposition 26. *For $k = 2$ the bounds in Lemma 25 are tight for all prime powers q .*

Proof. The proof follows from Definition 4, Proposition 7 and [2, Proposition 5.4]. \square

Remark 27. Let $\beta \in \mathbb{F}_q$ and let $c = (c_1, \dots, c_n)$ be a vector in $(\mathbb{F}_q)^n$. The number of coordinates of c equal to β is denoted by $c[\beta]$, namely $c[\beta] = |\{i \in \{1, \dots, n\} \mid c_i = \beta\}|$. In [2] the authors considered codes with the following property:

$$\text{There exists } \beta \in \mathbb{F}_q, \beta \neq 0, \text{ such that, for } a, b \in C, a[\beta] = b[\beta] \text{ only if } a = b. \quad (\text{A})$$

Due to [2, Corollary 5.2], if an $[n, k, d]_q$ MWS code C satisfy property (A), then $n \geq \lceil \frac{q \cdot q_k + 1}{2} \rceil$. This follows by applying the bound in the Lemma 25 to the $[2n + 1, k + 1, d']_q$ MWS code \bar{C} arisen from the construction given in [2, Proposition 4.1]. Using the same strategy we can get

$$\Delta(c) \geq \frac{(q-2)(q^k - q)(q^{k+1} - 1)}{4(q-2)^2 \cdot q}.$$

In this setting the q_k smallest elements in $S(\bar{C})$ are exactly the elements in $S(C)$. So the minimum distance of the new $[2n + 1, k + 1, d']_q$ MWS code \bar{C} coincides with the minimum distance of $[n, k, d]_q$ code C which means $d' = d$.

Proposition 28. *There exists an $[7, 3]_2$ strictly compact MWS code C , and there exists an $[32, 3]_3$ MWS code C' which is not strictly compact.*

Proof. The generator matrix of an $[7, 3]_2$ is a matrix $G \in \mathbb{F}_2^{3 \times 7}$ where the (multi)set of columns M can be generated by 3 linearly independent vectors v_1, v_2, v_3 in \mathbb{F}_2^3 where $m(v_i) = 2^i$. \mathbb{F}_2^3 contains $q_3 = 7$ hyperplanes (2-dimensional subspaces) with characters $\{0, 1, \dots, 6\}$. Due to Remark 23, the set of nonzero weights is $S(C) = \{1, 2, \dots, 7\}$. It is easy to verify that $d = 1$, $\Delta(C) = 0$, $|S(C)| = q_k$ and $n \in S(C)$. So C is a strictly compact MWS code. The existence of $[7, 3]_2$ MWS code is also shown in [11, 2].

The second part was also proven in [2]. Using the set of characters of hyperplanes, we can determine $S(C') = \{10, 14, 16, 19, 20, 21, 22, 24, 21, 20, 26, 27, 28, 30, 31\}$, $d = 10$ and $\Delta(C') = 50$. All the parameters satisfy the bounds given in the Lemma 25. Moreover, we already proved that there is no strictly compact MWS code when $q.k$ is odd. \square

Proposition 29. *For each $k \geq 2$ there exists an MWS code C of length, minimum distance and spread*

$$n = q_{k-1} \binom{q_k}{2}, \quad d = q^{k-2} \left[\binom{q_k - 1}{2} - q_k + 1 \right] \quad \text{and} \quad \Delta(C) = \frac{q_k \cdot q_{k-1} (q_k \cdot q_{k-1} - q_{k-1} - q)}{2}.$$

Proof. Let $\{H_0, \dots, H_{q_k-1}\}$ be the set of hyperplanes in $(\mathbb{F}_q)^k$. Define the generating matrix G as follows. For each vector $v \in \mathbb{F}_q^k$, let $m(v) = \sum_{v \in H_i} i$. For $k = 2$, a hyperplane is just a single vector of length k over \mathbb{F}_q , so two hyperplanes might coincide or disjoint. If $k = 3$, then two distinct hyperplanes have intersection in a 1-dimensional subspace. So for $k \geq 3$, a pair of distinct hyperplanes have intersection in a $k - 2$ -subspace. As a result, for $k \geq 2$ and $0 \leq s \leq \frac{q^k - 1}{q - 1} - 1 = q_k - 1$ we have

$$\text{Char}_G(H_s) = q_{k-2} \binom{q_k}{2} + (q_{k-1} - q_{k-2}) \cdot s,$$

which tells the minimum distance is $d = n - \text{Char}_G(H_{q_k-1}) = q^{k-2} \left[\binom{q_k - 1}{2} - q_k + 1 \right]$ and the number of columns of G should be $n = q_{k-1} \binom{q_k}{2}$. The rest follows by applying Theorem 13. \square

The above length was given in [1, Proposition 3.3] and in [2, Corollary 5.9] the existence of MWS codes with length $n < q^{\frac{k^2+k-4}{2}}$ and dimension $k \geq 3$ is proven. The later gives a shorter length than the length in Proposition 29 where $k = 3$.

7 Conclusions

In this work we introduce the notion of spread of a code, a tool to study the fundamental parameters of a code w.r.t the weight distribution of a target code. More precisely, the spread is a measure of how much the weight distribution of a code C is distant from the weight distribution of a target code. We focus here on MWS codes, a class of codes studied in the past few years by several authors, and we apply our methods to study the parameters of known examples of MWS codes. As a result of our approach, we are able to completely characterise the length of MWS codes according to their spread and to provide bounds on their minimum distances (Proposition 18). Moreover, we specialise

our results to two sub-families, namely to compact (Corollary 19) and strictly-compact (Theorem 10) MWS codes. We believe that the results obtained for MWS codes are a hint for the usefulness of analysing the parameters of families of codes according to their spread.

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