A Construction Framework for Mutually Orthogonal Complementary Sequence Sets with Flexible Lengths

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Abstract

Owing to the beautiful aperiodic correlation sum properties, mutually orthogonal complementary sequence sets (MOCSSs) are widely used in communication and radar. Known MOCSSs mostly have length of a power of an integer. This limits the application of MOCSSs in practical multi-carrier code division multiple access (MC-CDMA) systems with varying numbers of subcarriers. Recently, a construction of MOCSSs with non-power-of-two lengths was proposed by Wu *et al.* (IEEE Trans. Inf. Theory, 2020). The objective of this paper is to develop a general framework to construct more MOCSSs with flexible lengths. The proposed framework is based on complete complementary codes and even-shift complementary sequence sets.

1 Introduction

A pair of sequences is called Golay complementary pair (GCP) if its aperiodic autocorrelation sums are zero for except at zero time shifts [1]. The either constituent sequence in a GCP is called a Golay complementary sequence. Such sequences have found have many applications, please refer to [2] for details. However, the lengths of GCPs are limited, for example, the lengths of known binary GCPs are even number of the form $2^a 10^b 26^c$, where a, b, c are all non-negative integers. Therefore, Tseng and Liu proposed complementary sequence sets (CSSs) as an extension of GCPs in 1972 [3]. Later, CSSs also have found have important applications in many scenes [4]-[6]. Furthermore, in [3], the authors also proposed the concept of mutually orthogonal CSSs (MOCSSs), which a family of sets contains multiple CSSs, and the sums of aperiodic cross-correlation are zero between any two distinct sets for all time shifts. In 1988, Suehiro and Hatori presented complete complementary codes (CCCs) whose family size M achieves the upper bound of MOCSSs (i.e., $M \leq N$, where N is the number of constituent sequences in a CSS, called the block size) [7]. Due to the correlation properties, MOCSSs can achieve zero multipath interference (MPI) and zero multi-user interference (MUI) performance [8]. For this reason, they have been used as a key component of multi-carrier code division multiple access (MC-CDMA) [9]. In addition, MOCSSs also have some important applications in other aspects, such as multiple-input and multiple-output (MIMO) radar [10], optimal channel estimation in MIMO frequency-selective channels [11], etc.

How to construct MOCSSs is a high-profile academic direction. In [3], the authors proposed some constructions of MOCSSs based on a series of sequence operations. But the lengths of MOCSSs obtained by these methods are limited. In recent years, the tools for studying MOCSSs have gradually increased, such as concatenating iteration [12]-[13], unitary matrices [7], [14]-[15], Hadamard matrices [16]-[17], generalized Boolean functions [18]-[21] and paraunity matrices [22]-[24]. It is worthy to point that the lengths of these MOCSSs are mostly the power of an integer. Although MOCSSs have been proposed with flexible lengths in [25]-[26], these MOCSSs contain multiple null symbols. Recently, Wu, Chen, and Liu [21] proposed a generic construction of MOCSSs with length $2^m + 2^v$ based on generalized Boolean functions. To meet the requirements of different communications systems, it is interesting topic to construct more MOCSSs with non-power of an integer lengths. The motivation of this paper is to propose a construction of MOCSSs with non-power of an integer lengths, based on the inherent properties of CCCs and even-shift complementary sequence sets (ESCSSs). Furthermore, the constructions of ESCSSs have been studied.

The rest of this paper is organized as follows. In Section II, we give some definitions and some notations. In Sections III and IV, we propose a novel construction of MOCSSs based on ESCSSs, and also give several constructions of ESCSSs. Finally, we summarize the whole paper in Section V.

2 Preliminaries

Let us define some notations that are used uniformly throughout this paper.

- $\xi = e^{\sqrt{-1}\frac{2\pi}{q}}$ denotes the primitive complex q-th root of unity;
- $\mathbb{Z}_q = \{0, 1, \cdots, q-1\}$ is the set of integers modulo q;
- For $x \in \mathbb{Z}_q$, $-x$ denotes the additive inverse of x;
- x^* denotes the conjugate of complex number x;
- $\oplus$ denotes the modulo q addition;
- $-\mathbf{a}$ denotes the negation of $\mathbf{a}$;
- $\overleftarrow{\mathbf{a}}$ denotes the denotes the reverse of $\mathbf{a}$;
- $\mathbf{a}||\mathbf{b}$ denotes the horizontal concatenation of $\mathbf{a}$ and $\mathbf{b}$;
- $\mathbf{A} \bigcup \mathbf{B}$ denotes the union of two sequence sets $\mathbf{A}$ and $\mathbf{B}$.

2.1 Correlation Function

A sequence $\mathbf{a} = (a_0, a_1, \cdots, a_{L-1})$ is called a q-ary sequence if $a_i \in \mathbb{Z}_q$ for all $i \in \{0, 1, \cdots, L-1\}$. Unless otherwise specified, the sequences of this paper are defined over $\mathbb{Z}_q$. Given two length-L sequences $\mathbf{a}$ and $\mathbf{b}$, their aperiodic cross-correlation function (ACCF) is defined as

$$R_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{i=0}^{L-1-\tau} \xi^{a_i - b_{i+\tau}},\ 0 \le \tau \le L-1. \tag{1}$$

When $\mathbf{a} = \mathbf{b}$, $R_{\mathbf{a},\mathbf{b}}(\tau)$ is called the aperiodic auto-correlation function (AACF). For simplicity, the AACF of $\mathbf{a}$ will be sometimes written as $R_{\mathbf{a}}(\tau)$.

Let $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\}$ and $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_N\}$ be two sequence sets of length L. The ACCF between $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$R_{\mathbf{A},\mathbf{B}}(\tau) = \sum_{n=1}^{N} R_{\mathbf{a}_n,\mathbf{b}_n}(\tau),\ 0 \le \tau \le L-1. \tag{2}$$

Similarly, when $\mathbf{A} = \mathbf{B}$, $R_{\mathbf{A},\mathbf{B}}(\tau)$ is called the AACF. For convenience, the AACF of $\mathbf{A}$ is abbreviated as $R_{\mathbf{A}}(\tau)$.

2.2 Mutually Orthogonal Complementary Sequence Sets

Definition 1. A sequence set $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\}$ of length L is called a complementary sequence set (CSS), denoted by (N, L)-CSS, if

$$R_{\mathbf{A}}(\tau) = 0,\ \text{for all } \tau \ne 0. \tag{3}$$

Definition 2. Let $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_M\}$ be a family of M sequence sets, where $\mathbf{A}_m$ contains N sequences of length L. The family $\mathcal{A}$ is called a mutually orthogonal complementary sequence set (MOCSS), denoted by (M, N, L)-MOCSS, if

$$R_{\mathbf{A}_i,\mathbf{A}_j}(\tau) = \begin{cases} NL, & i = j, \tau = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 3. *For an (M, N, L)-MOCSS, the upper bound of family size satisfies*

$$M \le N. \tag{5}$$

When $M = N$, the MOCSS is called a complete complementary code (CCC) [15].

Definition 4. A sequence set $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_N\}$ of length L is called an even-shift complementary sequence set (ESCSS), denoted by (N, L)-ESCSS, if $R_{\mathbf{B}}(2k) = 0$ for all $1 \le k < L/2$.

2.3 Interleaving Operation

For two sequences $\mathbf{a}$ and $\mathbf{b}$ of length L, define an interleaved sequence $\mathbf{c}$ of length $2L$ as follows:

$$\mathbf{c} = I(\mathbf{a}, \mathbf{b}) = (a_0, b_0, a_1, b_1, \cdots, a_{L-1}, b_{L-1}),$$

where I is called the interleaving operator. It can be observed that

$$R_{\mathbf{c}}(\tau) = \begin{cases} R_{\mathbf{a}}(k) + R_{\mathbf{b}}(k), & \tau = 2k; \\ R_{\mathbf{a},\mathbf{b}}(k) + R_{\mathbf{b},\mathbf{a}}(k+1), & \tau = 2k+1. \end{cases} \tag{6}$$

3 A Construction of MOCSSs

In this section, we will give the main results of this paper. To facilitate the description, we define an operator φ before giving the main results. Let $\mathbf{a}$ be a sequence of length L_1, $\mathbf{b}$ and $\mathbf{c}$ be two sequences with same length L_2. The sequence $\mathbf{s} = \varphi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is defined as follows

$$\begin{aligned} \mathbf{s} &= \varphi(\mathbf{a}, \mathbf{b}, \mathbf{c}) \\ &= \begin{cases} \big((a_0 \oplus \mathbf{b})||(a_1 \oplus \mathbf{c})||(a_2 \oplus \mathbf{b})|| \cdots ||(a_{L_1-2} \oplus \mathbf{b})||(a_{L_1-1} \oplus \mathbf{c})\big), & L_2 \text{ even}; \\ \big((a_0 \oplus \mathbf{b})||(a_1 \oplus \mathbf{c})||(a_2 \oplus \mathbf{b})|| \cdots ||(a_{L_1-2} \oplus \mathbf{c})||(a_{L_1-1} \oplus \mathbf{b})\big), & L_2 \text{ odd}. \end{cases} \end{aligned} \tag{7}$$

Let $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_N\}$ and $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ be two sequence sets with same length L_2. The sequence set $\mathbf{S} = \varphi(\mathbf{a}, \mathbf{B}, \mathbf{C})$ is a sequence set consisting of N sequences of length L_1L_2 defined as follows

$$\begin{aligned} \mathbf{S} &= \varphi(\mathbf{a}, \mathbf{B}, \mathbf{C}) \\ &= \{\varphi(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_1), \varphi(\mathbf{a}, \mathbf{b}_2, \mathbf{c}_2), \cdots, \varphi(\mathbf{a}, \mathbf{b}_N, \mathbf{c}_N)\}. \end{aligned} \tag{8}$$

Theorem 5. *Let $\mathbf{P}$, $\mathbf{Q}$ be two (N, L_1)-ESCSSs and $\mathcal{A}$ be an (M, M, L_2)-CCC. Let*

$$\mathbf{S}_i = \bigcup_{n=1}^{N} \big\{\varphi(\mathbf{p}_n, \mathbf{A}_{2i-1}, \mathbf{A}_{2i})\big\}, \tag{9}$$

$$\mathbf{S}_{i+\frac{M}{2}} = \bigcup_{n=1}^{N} \big\{\varphi(\mathbf{q}_n, \mathbf{A}_{2i-1}, \mathbf{A}_{2i})\big\}, \tag{10}$$

where $i = 1, 2, \cdots, \frac{M}{2}$. If $R_{\mathbf{P},\mathbf{Q}}(\tau) = 0$ and $R_{\mathbf{Q},\mathbf{P}}(\tau) = 0$ for all $0 \le \tau \le L_1 - 1$, then the family of sets $\mathcal{S} = \{\mathbf{S}_1, \mathbf{S}_2, \cdots, \mathbf{S}_M\}$ is an (M, MN, L_1L_2)-MOCSS.

Proof. We only prove that L_1 is even, the case that L_1 is odd can be similarly discussed.

Let $\tau = kL_2 + t$, where $0 \le k \le L_1 - 1$, $0 \le t \le L_2 - 1$. For any $x, y \in \{1, 2, \cdots, M\}$, we need to prove that

$$R_{\mathbf{S}_x,\mathbf{S}_y}(\tau) = \begin{cases} MNL_1L_2, & x = y, k = t = 0; \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

Let $I_1 = \{1, 2, \cdots, \frac{M}{2}\}$ and $I_2 = \{\frac{M}{2}+1, +\frac{M}{2}+2, \cdots, M\}$. We divide the proof into the following four cases: **Case 1**: $x, y \in I_1$; **Case 2**: $x, y \in I_2$; **Case 3**: $x \in I_1$ and $y \in I_2$; **Case 4**: $y \in I_1$ and $x \in I_2$. In fact, we only prove Case 1 and Case 3. The other two cases can be similarly proved.

For Case 1, when k is an even number, the ACCF of $\mathbf{S}_x$ and $\mathbf{S}_y$ can be written as

$$\begin{aligned}
R_{\mathbf{S}_x,\mathbf{S}_y}(\tau) =& \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} R_{\mathbf{a}_{2x-1,m},\mathbf{a}_{2y-1,m}}(t)\xi^{p_n^{2i}-p_n^{2i+k}} \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} R_{\mathbf{a}_{2x,m},\mathbf{a}_{2y,m}}(t)\xi^{p_n^{2i+1}-p_n^{2i+k+1}} \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} R^*_{\mathbf{a}_{2y,m},\mathbf{a}_{2x-1,m}}(L_2-t)\xi^{p_n^{2i}-p_n^{2i+k+1}} \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=1}^{\frac{L_1-k-2}{2}} R^*_{\mathbf{a}_{2y-1,m},\mathbf{a}_{2x,m}}(L_2-t)\xi^{p_n^{2i-1}-p_n^{2i+k}} \\
=& R_{\mathbf{A}_{2x-1},\mathbf{A}_{2y-1}}(t)\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} \xi^{p_n^{2i}-p_n^{2i+k}} \\
&+ R_{\mathbf{A}_{2x},\mathbf{A}_{2y}}(t)\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} \xi^{p_n^{2i+1}-p_n^{2i+k+1}} \\
=& \begin{cases} ML_2 \cdot R_{\mathbf{P}}(k), & x=y, t=0; \\ 0, & \text{otherwise.} \end{cases} \\
=& \begin{cases} ML_2NL_1, & x=y, k=t=0; \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where the last equality is due to the fact that $\mathbf{P}$ is an ESCSS. When k is an odd number, we can prove it similarly. Hence the first case is proved.

For Case 3, when k is an even number, we have

$$\begin{aligned}
R_{\mathbf{S}_x,\mathbf{S}_y}(\tau) =& \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} R_{\mathbf{a}_{2x-1,m},\mathbf{a}_{2z-1,m}}(t)\xi^{p_n^{2i}-q_n^{2i+k}} \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} R_{\mathbf{a}_{2x,m},\mathbf{a}_{2z,m}}(t)\xi^{p_n^{2i+1}-q_n^{2i+k+1}} \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}} R^*_{\mathbf{a}_{2z,m},\mathbf{a}_{2x-1,m}}(L_2-t)\xi^{p_n^{2i}-q_n^{2i+k+1}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{i=1}^{\frac{L_1-k-2}{2}} R^*_{\mathbf{a}_{2z-1,m},\mathbf{a}_{2x,m}}(L_2-t)\xi^{p_n^{2i-1}-q_n^{2i+k}}\\
=&R_{\mathbf{A}_{2x-1},\mathbf{A}_{2z-1}}(t)\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}}\xi^{p_n^{2i}-q_n^{2i+k}}\\
&+R_{\mathbf{A}_{2x},\mathbf{A}_{2z}}(t)\sum_{n=1}^{N}\sum_{i=0}^{\frac{L_1-k-2}{2}}\xi^{p_n^{2i+1}-q_n^{2i+k+1}}\\
=&\begin{cases} ML_2\cdot R_{\mathbf{P},\mathbf{Q}}(k), & x=z, t=0;\\ 0, & \text{otherwise.}\end{cases}\\
=&\,0,
\end{aligned}
$$

where $z = y - \frac{M}{2}$. We can prove it similarly when k is an odd number.

In summary, the family of sets $\mathcal{S}$ is an (M, NM, L_1L_2)-MOCSSs. □

Remark 6. Actually, it is easy to find two sequence sets $\mathbf{P}$ and $\mathbf{Q}$ satisfy the conditions $R_{\mathbf{P},\mathbf{Q}}(\tau) = 0$ and $R_{\mathbf{Q},\mathbf{R}}(\tau) = 0$. i.e.,

$$
\begin{aligned}
&\mathbf{P} = \{\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_N\},\\
&\mathbf{Q} = \{-\overleftarrow{\mathbf{p}_2}, -\overleftarrow{\mathbf{p}_1} \oplus \frac{q}{2}, \cdots, -\overleftarrow{\mathbf{p}_N}, -\overleftarrow{\mathbf{p}_{N-1}} \oplus \frac{q}{2}\}.
\end{aligned} \tag{12}
$$

In the following, we will give an example to illustrate Theorem 5.

Example 7. Let $q = 2$, $\mathbf{P}$ and $\mathbf{Q}$ be two $(2, 11)$-ESCSSs, i.e.,

$$
\mathbf{P} = \left\{\begin{array}{c}\mathbf{p}_1\\ \mathbf{p}_2\end{array}\right\} = \left\{\begin{array}{c}(0,0,0,0,0,0,0,1,1,1,0)\\ (0,0,1,1,0,0,0,0,0,1,1)\end{array}\right\},
$$

$$
\mathbf{Q} = \left\{\begin{array}{c}\mathbf{q}_1\\ \mathbf{q}_2\end{array}\right\} = \left\{\begin{array}{c}(1,1,0,0,0,0,0,1,1,0,0)\\ (1,0,0,0,1,1,1,1,1,1,1)\end{array}\right\}.
$$

Let $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ be a $(4, 4, 1)$-CCC, i.e.,

$$
\mathbf{A}_1 = \left\{\begin{array}{c}\mathbf{a}_{1,1}\\ \mathbf{a}_{1,2}\\ \mathbf{a}_{1,2}\\ \mathbf{a}_{1,2}\end{array}\right\} = \left\{\begin{array}{c}(0)\\(0)\\(0)\\(0)\end{array}\right\}, \mathbf{A}_2 = \left\{\begin{array}{c}\mathbf{a}_{2,1}\\ \mathbf{a}_{2,2}\\ \mathbf{a}_{2,2}\\ \mathbf{a}_{2,2}\end{array}\right\} = \left\{\begin{array}{c}(0)\\(1)\\(0)\\(1)\end{array}\right\},
$$

$$
\mathbf{A}_3 = \left\{\begin{array}{c}\mathbf{a}_{3,1}\\ \mathbf{a}_{3,2}\\ \mathbf{a}_{3,2}\\ \mathbf{a}_{3,2}\end{array}\right\} = \left\{\begin{array}{c}(0)\\(0)\\(1)\\(1)\end{array}\right\}, \mathbf{A}_4 = \left\{\begin{array}{c}\mathbf{a}_{4,1}\\ \mathbf{a}_{4,2}\\ \mathbf{a}_{4,2}\\ \mathbf{a}_{4,2}\end{array}\right\} = \left\{\begin{array}{c}(0)\\(1)\\(1)\\(0)\end{array}\right\}.
$$

According to Theorem 5, we obtain a $(4, 8, 11)$-MOCSS $\mathcal{S} = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4\}$ as follows:

$$\mathbf{S}_1 = \begin{Bmatrix} \mathbf{s}_{1,1} \\ \mathbf{s}_{1,2} \\ \mathbf{s}_{1,3} \\ \mathbf{s}_{1,4} \\ \mathbf{s}_{1,5} \\ \mathbf{s}_{1,6} \\ \mathbf{s}_{1,7} \\ \mathbf{s}_{1,8} \end{Bmatrix} = \begin{Bmatrix} (0,0,0,0,0,0,0,1,1,1,0) \\ (0,0,1,1,0,0,0,0,0,1,1) \\ (0,1,0,1,0,1,0,0,1,0,0) \\ (0,1,1,0,0,1,0,1,0,0,1) \\ (0,0,0,0,0,0,0,1,1,1,0) \\ (0,0,1,1,0,0,0,0,0,1,1) \\ (0,1,0,1,0,1,0,0,1,0,0) \\ (0,1,1,0,0,1,0,1,0,0,1) \end{Bmatrix},$$

$$\mathbf{S}_2 = \begin{Bmatrix} \mathbf{s}_{2,1} \\ \mathbf{s}_{2,2} \\ \mathbf{s}_{2,3} \\ \mathbf{s}_{2,4} \\ \mathbf{s}_{2,5} \\ \mathbf{s}_{2,6} \\ \mathbf{s}_{2,7} \\ \mathbf{s}_{2,8} \end{Bmatrix} = \begin{Bmatrix} (0,0,0,0,0,0,0,1,1,1,0) \\ (0,0,1,1,0,0,0,0,0,1,1) \\ (0,1,0,1,0,1,0,0,1,0,0) \\ (0,1,1,0,0,1,0,1,0,0,1) \\ (1,1,1,1,1,1,1,0,0,0,1) \\ (1,1,0,0,1,1,1,1,1,0,0) \\ (1,0,1,0,1,0,1,1,0,1,1) \\ (1,0,0,1,1,0,1,0,1,1,0) \end{Bmatrix},$$

$$\mathbf{S}_3 = \begin{Bmatrix} \mathbf{s}_{3,1} \\ \mathbf{s}_{3,2} \\ \mathbf{s}_{3,3} \\ \mathbf{s}_{3,4} \\ \mathbf{s}_{3,5} \\ \mathbf{s}_{3,6} \\ \mathbf{s}_{3,7} \\ \mathbf{s}_{3,8} \end{Bmatrix} = \begin{Bmatrix} (1,1,0,0,0,0,0,1,1,0,0) \\ (1,0,0,0,1,1,1,1,1,1,1) \\ (1,0,0,1,0,1,0,0,1,1,0) \\ (1,1,0,1,1,0,1,0,1,0,1) \\ (1,1,0,0,0,0,0,1,1,0,0) \\ (1,0,0,0,1,1,1,1,1,1,1) \\ (1,0,0,1,0,1,0,0,1,1,0) \\ (1,1,0,1,1,0,1,0,1,0,1) \end{Bmatrix},$$

$$\mathbf{S}_4 = \begin{Bmatrix} \mathbf{s}_{4,1} \\ \mathbf{s}_{4,2} \\ \mathbf{s}_{4,3} \\ \mathbf{s}_{4,4} \\ \mathbf{s}_{4,5} \\ \mathbf{s}_{4,6} \\ \mathbf{s}_{4,7} \\ \mathbf{s}_{4,8} \end{Bmatrix} = \begin{Bmatrix} (1,1,0,0,0,0,0,1,1,0,0) \\ (1,0,0,0,1,1,1,1,1,1,1) \\ (1,0,0,1,0,1,0,0,1,1,0) \\ (1,1,0,1,1,0,1,0,1,0,1) \\ (0,0,1,1,1,1,1,0,0,1,1) \\ (0,1,1,1,0,0,0,0,0,0,0) \\ (0,1,1,0,1,0,1,1,0,0,1) \\ (0,0,1,0,0,1,0,1,0,1,0) \end{Bmatrix}.$$

4 Constructions of ESCSSs

Our proposed MOCSSs are based on CCCs and ESCSSs. The lengths of CCCs are limited, so we will discuss the constructions of ESCSSs in this section.

Theorem 8. *Let* $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_N\}$ *be an* (N, L)*-CSS and let*

$$\begin{aligned}\mathbf{B} &= \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{N-1}, \mathbf{b}_N\} \\ &= \left\{I(\mathbf{a}_1, \mathbf{a}_2), I(\mathbf{a}_1, \mathbf{a}_2 \oplus \frac{q}{2}), \cdots, I(\mathbf{a}_{N-1}, \mathbf{a}_N), I(\mathbf{a}_{N-1}, \mathbf{a}_N \oplus \frac{q}{2})\right\}\end{aligned}$$

be a sequence set of length $2L$*. Then* $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ *given by Table 1 is an* $(N, 2L+1)$*-ESCSS.*

Table 1: Two Cases of ESCSSs of Length $2L+1$ from CSSs

Case	$\mathbf{c}_1$	$\mathbf{c}_2$	$\cdots$	$\mathbf{c}_{N-1}$	$\mathbf{c}_N$
1	$\mathcal{I}(\mathbf{b}_1, d_1, 0)$	$\mathcal{I}(\mathbf{b}_2, d_1, 0)$	$\cdots$	$\mathcal{I}(\mathbf{b}_{N-1}, d_{N/2}, 0)$	$\mathcal{I}(\mathbf{b}_N, d_{N/2}, 0)$
2	$\mathcal{I}(\mathbf{b}_1, d_1, 2L)$	$\mathcal{I}(\mathbf{b}_1, d_1 \oplus \frac{q}{2}, 2L)$	$\cdots$	$\mathcal{I}(\mathbf{b}_N, d_{N/2}, 2L)$	$\mathcal{I}(\mathbf{b}_N, d_{N/2} \oplus \frac{q}{2}, 2L)$

where $d_j \in \mathbb{Z}_q$, $j \in \{1, 2, \cdots, N/2\}$.

Proof. We will prove Case 1, and Case 2 can be similarly proved.

Let $\tau = 2k$ with $k > 0$. According to Eq. (6), the AACF of $\mathbf{c}_j$ can be expressed as

$$R_{\mathbf{c}_j}(\tau) = \begin{cases} R_{\mathbf{a}_j}(k) + R_{\mathbf{a}_{j+1}}(k) + \xi^{d_{(j+1)/2} - a_{j+1}^{k-1}}, & j \text{ is odd;} \\ R_{\mathbf{a}_{j-1}}(k) + R_{\mathbf{a}_j}(k) - \xi^{d_{j/2} - a_j^{k-1}}, & j \text{ is even.} \end{cases} \tag{13}$$

Then we have $R_{\mathbf{C}}(\tau) = 2R_{\mathbf{A}}(k) = 0$. This completes the proof. □

Theorem 9. *Let* $\mathbf{A}$ *be an* (N, L_1)*-ESCSS and* $\mathbf{B}$ *be an* (N, L_2)*-ESCSS. Then the sequence set* $\mathbf{C}$

$$\begin{aligned} \mathbf{C} = \Big\{ & (\mathbf{a}_1 || \mathbf{b}_1), (\mathbf{a}_2 || \mathbf{b}_2), \cdots, (\mathbf{a}_N || \mathbf{b}_N), \\ & (\mathbf{a}_1 || \mathbf{b}_1 \oplus \frac{q}{2}), (\mathbf{a}_2 || \mathbf{b}_2 \oplus \frac{q}{2}), \cdots, (\mathbf{a}_N || \mathbf{b}_N \oplus \frac{q}{2}) \Big\}. \end{aligned} \tag{14}$$

is a $(2N, L_1 + L_2)$*-ESCSS.*

Proof. Let $\mathbf{c}_{1,i} = (\mathbf{a}_i || \mathbf{b}_i)$, $\mathbf{c}_{2,i} = (\mathbf{a}_i || \mathbf{b}_i \oplus \frac{q}{2})$ and $\mathbf{C} = \{\mathbf{c}_{1,i}, \mathbf{c}_{2,i} : i = 1, 2, \cdots, N\}$. Without loss of generality, we assume that $L_1 \leq L_2$. For any even integer τ with $1 \leq \tau < L_1 + L_2$, we have

$$R_{\mathbf{c}_{1,i}}(\tau) = \begin{cases} R_{\mathbf{a}_i}(\tau) + R_{\mathbf{b}_i}(\tau) + R_{\mathbf{b}_i, \mathbf{a}_i}(L_1 - \tau), & 1 \leq \tau \leq L_1 - 1; \\ R_{\mathbf{b}_i}(\tau) + R_{\mathbf{a}_i, \mathbf{b}_i}(\tau - L_1), & L_1 \leq \tau \leq L_2 - 1; \\ R_{\mathbf{a}_i, \mathbf{b}_i}(\tau - L_1), & L_2 \leq \tau \leq L_1 + L_2 - 1; \end{cases} \tag{15}$$

and

$$R_{\mathbf{c}_{2,i}}(\tau) = \begin{cases} R_{\mathbf{a}_i}(\tau) + R_{\mathbf{b}_i}(\tau) - R_{\mathbf{b}_i, \mathbf{a}_i}(L_1 - \tau), & 1 \leq \tau \leq L_1 - 1; \\ R_{\mathbf{b}_i}(\tau) - R_{\mathbf{a}_i, \mathbf{b}_i}(\tau - L_1), & L_1 \leq \tau \leq L_2 - 1; \\ -R_{\mathbf{a}_i, \mathbf{b}_i}(\tau - L_1), & L_2 \leq \tau \leq L_1 + L_2 - 1. \end{cases} \tag{16}$$

Hence one has

$$R_{\mathbf{C}}(\tau) = \sum_{i=1}^{N} (R_{\mathbf{c}_{1,i}}(\tau) + R_{\mathbf{c}_{2,i}}(\tau))$$

$$= \begin{cases} 2[R_{\mathbf{A}}(\tau) + R_{\mathbf{B}}(\tau)], & 1 \le \tau \le L_1 - 1; \\ 2R_{\mathbf{B}}(\tau), & L_1 \le \tau \le L_2 - 1; \\ 0, & L_2 \le \tau \le L_1 + L_2 - 1; \end{cases}$$
$$= 0.$$

Therefore, the set $\mathbf{C}$ is an ESCSS. This completes the proof. □

Remark 10. In Theorem 9, if $\mathbf{A}$ is an (N, L_1)-CSS and $\mathbf{B}$ is an (N, L_2)-CSS, then $\mathbf{C}$ is an $(2N, L_1 + L_2)$-CSS.

Theorem 11. *Let* $\mathbf{P}$*,* $\mathbf{Q}$ *be two* (N_1, L_1)*-ESCSSs and* $\mathbf{A}$ *be an* (N_2, L_2)*-ESCSS. Then the sequence set*

$$\mathbf{S} = \bigcup_{n=1}^{N_2} \left\{\varphi(\mathbf{a}_n, \mathbf{P}, \mathbf{Q})\right\} \tag{17}$$

is an (N_1N_2, L_1L_2)*-ESCSS if* $R_{\mathbf{P},\mathbf{Q}}(\tau) = 0$ *and* $R_{\mathbf{Q},\mathbf{R}}(\tau) = 0$ *for all* τ*.*

Here we omit the proof of this theorem because it is similar to Theorem 5.

Example 12. Taking $q = 4$, let

$$\mathbf{P} = \left\{(1, 3, 0, 0, 0), (0, 0, 2, 1, 1)\right\},$$
$$\mathbf{Q} = \left\{(3, 3, 2, 0, 0), (2, 2, 2, 3, 1)\right\}$$

be two $(2, 5)$-ESCSSs and $\mathbf{A} = \left\{(2, 0, 1), (2, 1, 3)\right\}$ be a $(2, 3)$-ESCSS. Then we obtain an $(4, 15)$-ESCSS $\mathbf{S}$ by using Theorem 11. i.e.,

$$\mathbf{S} = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\}$$
$$= \left\{ \begin{array}{c} (3, 1, 2, 2, 2, 3, 3, 2, 0, 0, 2, 0, 1, 1, 1) \\ (2, 2, 0, 3, 3, 2, 2, 2, 3, 1, 1, 1, 3, 2, 2) \\ (2, 2, 0, 3, 3, 3, 3, 3, 0, 2, 3, 3, 1, 0, 0) \\ (3, 1, 2, 2, 2, 0, 0, 3, 1, 1, 0, 2, 3, 3, 3) \end{array} \right\}.$$

The even shift AACF of $\mathbf{S}$ is

$$\Big(R_{\mathbf{S}}(\tau)\Big)_{\tau \equiv 0 (mod\ 2)}^{14} = (60, \mathbf{0}_7).$$

In Table 2, we list binary ESCSSs with set size 2 and 4 of length $L \le 50$. In fact, the binary ESCSSs with size 4 of length $L \le 500$ can be constructed by existing methods. In other words, the $(M, 4M)$-MOCSSs of length $L \le 500$ can be constructed by using our proposed framework, where M the family size of CCCs.

Table 2: Binary ESCSSs for Various Lengths

Size	Lengths
2	3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 17, 18, 20, 21, 22, 24, 26, 28, 30, 32, 33, 34, 36, 40, 41, 42, 44, 48, 50
4	3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50

Table 3: Parameters of Binary (M, N)-MOCSSs

Ref. \ Length	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	M/N
[15], [18]		✓				✓								✓					4/4
						✓								✓					8/8
														✓					16/16
[21]		✓		✓		✓		✓		✓				✓		✓		✓	2/4
						✓								✓				✓	4/8
														✓					8/16
				✓				✓		✓						✓		✓	2/8
										✓								✓	4/16
																			8/32
This Paper	✓	✓	✓	✓	✓	✓	✓	✓		✓		✓		✓	✓	✓		✓	2/4
	✓	✓	✓	✓	✓	✓	✓	✓		✓		✓		✓	✓	✓		✓	4/8
	✓	✓	✓	✓	✓	✓	✓	✓		✓		✓		✓	✓	✓		✓	8/16
	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	2/8
	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	4/16
	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	8/32

5 Conclusion

In this paper, we gave a construction of MOCSSs with various lengths based on CCCs and ESCSSs. Since the lengths of the known CCCs are limited, we proposed some constructions of ESCSSs. In Table 3, the binary MOCSSs of lengths between 3 and 20 are listed. Compared with MOCSSs from [15], [18] and [21], Theorem 5 can construct more MOCSSs with various lengths.

References

[1] M. J. E. Golay. Complementary series. *IRE Trans. Inf. Theory*, 7(2): 82–87, 1961.

[2] P. Z. Fan, M. Darnell. *Sequence Design for Communications Applications.* New York & Wiley, 1996.

[3] C. C. Tseng, C. L. Liu. Complementary sets of sequences. *IEEE Trans. Inf. Theory*, 18(5): 644–652, 1972.

[4] C. Y. Chen. Complementary sets of non-power-of-two length for peak-to-average power ratio reduction in OFDM. *IEEE Trans. Inf. Theory*, 62(12): 7538–7545, 2016.

[5] J. Aparicio, T. Shimura. Asynchronous detection and identification of multiple users by multi-carrier modulated complementary set of sequences. *IEEE Access*, 6: 22054–22069, 2018.

[6] H. Jeon, J. Lee, Y. Han, S. J. Kim, I. S. Kweon. Multi-image deblurring using complementary sets of fluttering patterns. *IEEE Trans. on Image Proc.*, 26(5): 2311–2326, 2017.

[7] N. Suehiro, M. Hatori. N-shift cross-orthogonal sequences. *IEEE Trans. Inf. Theory*, 34(1):143–146, 1988.

[8] H. H. Chen. *The Next Generation CDMA Techonolgies*. John Wiley & Sons, 2007.

[9] S. M. Tseng, M. R. Bell. Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences. *IEEE Trans. Commun.*, 48: 53–59, 2000.

[10] J. Tang, N. Zhang, Z. Ma, B. Tang. Construction of Doppler resilient complete complementary code in MIMO radar. *IEEE Trans. Signal Process.*, 62(18): 4704–4712, 2014.

[11] S. Q. Wang, A. Abdi. MIMO ISI channel estimation using uncorrelated Golay complementary sets. *IEEE Trans. Veh. Technol.*, 56(5): 3024–3039, 2007.

[12] C. Zhang, C. Han, Y. Liao, X. Lin, M. Hatori. Iterative method for constructing complete complementary sequences with lengths of $2^m N$. *Tsinghua Science and Technology*, 10(5): 605–609, 2005.

[13] Z. Y. Zhang, F. X. Zeng, G. X. Xuan. Mutually orthogonal sets of complementary sequences for multi-carrier CDMA systems. in Proc. *The 7th International Conference on Wireless Communications, Networking and Mobile Computings*, 1–4, 2010.

[14] C. G. Han, N. Suehiro, T. Hashimoto. A generation method for constructing $(N, N, MN/P)$ complete complementary sequences. in Proc. *Joint IST Workshop on Mobile Future and Symp. Trends in Commun.*, 70–73, 2004.

[15] C. G. Han, N. Suehiro, T. Hashimoto. A systematic framework for the construction of optimal complete complementary codes. *IEEE Trans. Inf. Theory*, 57(9): 6033–6042, 2011.

[16] X. J. Huang, Y. X. Li. Scalable complete complementary sets of sequences. in Proc. *IEEE Global Communications Conference*, 1056–1060, 2002.

[17] C. D. Marziani, J. Urena, M. Mazo, F. J. Alvarez, J. J. Garcia, Modular architecture for efficient generation and correlation of complementary set of sequences. *IEEE Trans. Signal Process.*, 55(5): 2323–2337, 2007.

[18] A. Rathinakumar, A. K. Chaturvedi. Complete mutually orthogonal Golay complementary sets from Reed-Muller Codes. *IEEE Trans. Inf. Theory*, 54(3): 1339–1346, 2008.

[19] C. Y. Chen, C. H. Wang, C. C. Chao. Complete complementary codes and generalized Reed-Muller codes. *IEEE Commun. Lett.*, 12(11): 849–851, 2008.

[20] Z. L. Liu, Y. L. Guang, U. Parampalli. New complete complementary codes for peak-to-mean power control in multi-carrier CDMA. *IEEE Trans. Commun.*, 26(3): 1105–1113, 2014.

[21] W. X. Wu, C. Y. Chen, Z. L. Liu. How to construct mutually orthogonal complementary sets with non-power-of-two lengths. *IEEE Trans. Inf. Theory*, Early Access Article, 2020.

[22] S. Das, S. Budišin, S. Majhi, Z. L. Liu, Y. L. Guan. A multiplier-free generator for polyphase complete complementary codes. *IEEE Trans. Signal Process.*, 66(5): 1184–1196, 2017.

[23] S. Das, S. Majhi, Z. L. Liu. A novel class of complete complementary codes and their applications for APU matrices. *IEEE Signal Process. Lett.*, 25(9): 1300–1304, 2018.

[24] S. Das, S. Majhi, S. Budišin, Z. L. Liu. A new construction framework for polyphase complete complementary codes with various lengths. *IEEE Trans. Signal Process.*, 67(10): 2639–2648, 2019.

[25] Y. J. Zhou, Y. Yang, Z. C. Zhou, K. Anand, S. Hu, Y. L. Guan. New complementary sets with low PAPR property under spectral null constraints. *Arxiv*, 2020.

[26] R. N. Ipanov, A. I. Baskakov, N. Olyunin, M. Ka. Radar signals with ZACZ based on pairs of D-code sequences and their compression algorithm. *IEEE Signal Process. Lett.*, 25(10): 1560–1564, 2018.

[27] B. S. Shen, Y. Yang, Z. C. Zhou. A construction of binary golay complementary sets based on even-shift complementary pairs. *IEEE Access*, 8: 29882–29890, 2020.